RECURSIVE COMPUTATION OF OPEN GROMOV-WITTEN INVARIANTS

ROI BLUMBERG

Let (X, ω) be a closed symplectic manifold, and let J be an ω-tame almost complex structure. Closed genus 0 Gromov-Witten invariants provide, intuitively, an enumeration of J-holomorphic curves that intersect a given list of representatives of homology classes of X. These invariants satisfy a list of basic properties, commonly called the Gromov-Witten axioms.

In 1994, Konsevitch and Manin proved that under the assumption that the cohomology ring of X is generated by divisors, the Gromov-Witten invariants of X can always be recursively computed from a set of initial values [9]. Assuming X is Fano, this set is always finite. These recursive computations rely on the initial values and the Gromov-Witten axioms. Special cases and extensions of these recursions were applied to enumerate, for example, rational curves in $\mathbb{C}P^n$ [9, 10] and in blowups of $\mathbb{C}P^n$ at points [3, 6].

Now, let $L \subset X$ be a closed, orientable, and relatively spin Lagrangian submanifold. In an analogous manner to the closed case, open Gromov-Witten invariants provide, intuitively, an enumeration of J-holomorphic disks with boundary in L, so that the interior of the disk intersects a list of representatives of homology classes of $X \backslash L$, and the boundary of the disk intersects a given number of representatives of the homology class of a point in L. Genus 0 open Gromov-Witten invariants were defined in several different works, such as [11], [2], [4], [12], and [13]. Under any definition, these invariants satisfy a set of basic properties, called the open Gromov-Witten axioms, that are analogous to the Gromov-Witten axioms.

It is natural to ask whether an analogous statement to Konsevitch and Manin's result can be made. Recursive computations in the spirit of Konsevitch and Manin were made in several special cases – for example, when L is the real locus of blowups of $\mathbb{C}P^2$ [1, 8], or the quadric surface $\mathbb{C}P^1 \times \mathbb{C}P^1$, or of the sixfold $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ [1], or of $\mathbb{C}P^n$ with odd n [5, 12]. Another such computation was made when L is the Chiang Lagrangian in $\mathbb{C}P^3$ [7].

In this work, a general statement that is analogous to Konsevitch and Manin's result is proven. Namely, it is shown that the open Gromov-Witten invariants of (X, L) can always be computed from a set of initial values if the cohomology ring of X is generated by divisors, and that this set can be even further restricted if one assume that there exists some special non-zero open Gromov-Witten invariant. Assuming X is Fano, this set is finite. The computations rely on the initial values and the open Gromov-Witten axioms.

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Email address: `roiblumberg@mail.tau.ac.il`

Tel Aviv University