RANK-STABILITY OF POLYNOMIAL EQUATIONS

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Ulam's stability problem asks when an approximate solution to a given equation, with respect to some metric, is 'close' to an exact solution. One famous case of this problem is due to Halmos [7], who asked whether two self-adjoint operators that almost commute are close to commuting self-adjoint operators. Here "almost commute" and "close" can be measures by different metrics, such as the operator norm or the Hilbert–Schmidt norm, and the answer depends on the chosen metric (see [8, 5, 2, 4]).

Stability of more general equations $w_1(\vec{x}) = \cdots = w_r(\vec{x}) = 1$ has been studied by Glebsky and Rivera [6], and for many natural choices of groups and metrics, it is a property of the group $\langle x_1, \ldots, x_d \mid w_1, \ldots, w_r \rangle$ rather than the specific presentation. In particular, stability of the commutation equation is just stability of abelian groups; finite groups are stable with respect to the Hamming distance on symmetric groups [6]. Group stability has been studied and connected to other group properties such as amenability and vanishing cohomology [1, 3].

In this talk, we will discuss a joint work with Tomer Bauer and Be'eri Greenfeld, in which we lay the foundations to study stability of polynomial equations on matrices. We set the following notation: for a matrix $A \in \mathrm{M}_n(F)$, we denote by $\widehat{A} \in \mathrm{M}_{\mathbb{N}}(F)$ the infinite matrix obtained by padding A with zero rows and columns.

Definition. *Let $\mathcal{A} = F\langle x_1, \ldots, x_d \rangle / \langle P_1, \ldots, P_r \rangle$ be a finitely presented algebra. We say that $\mathcal{A}$ is **rank-stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any d-tuple of $n \times n$ matrices $(A_1, \ldots, A_d)$ satisfying:*

$$\operatorname{rank} P_i(A_1, \ldots, A_d) < \delta n \quad \text{for all } 1 \leq i \leq r$$

there exists a solution $(B_1, \ldots, B_d)$ of $n' \times n'$ matrices of $P_1, \ldots, P_r$ such that

$$\operatorname{rank}(\widehat{A_i} - \widehat{B_i}) < \varepsilon n \quad \text{for all } 1 \leq i \leq d.$$

We also give a similar definition for Lie algebras, replacing the Lie bracket $[x, y]$ by the commutator $xy - yx$.

We show that while finite-dimensional associative algebras are rank-stable, "most" finite-dimensional Lie algebras are not; also, a group is rank-stable if and only if its group algebra is rank-stable. The rank-stability of $F[x_1, \ldots, x_n]$ is equivalent to the rank-stability of the commutating equation, and is still open; however, we show that rings that "behave like" polynomial rings can be non-rank-stable. Along the way, we prove that the rank-stability of an algebra does not depend on its presentation, which is surprisingly non-trivial.

We show that any rank-stable linear sofic algebra must be residually finite-dimensional; and indeed, the existence of finite-dimensional representations, even if not faithful, is crucial for some of our results. We also study stability under some natural constructions of algebra, and show the following results for algebras $\mathcal{A}, \mathcal{A}'$ which possess a finite-dimensional representation:

(1) $\mathcal{A}$ and $\mathcal{A}'$ are rank-stable if and only if the free product $\mathcal{A}*\mathcal{A}'$ is rank-stable, if and only if the direct product $\mathcal{A} \times \mathcal{A}'$ is rank-stable;
(2) For any $n \geq 1$, an algebra $\mathcal{A}$ is rank-stable if and only if $\mathrm{M}_n(\mathcal{A})$ is rank-stable.

To study the above questions we present a compression machinery, which allows us to effectively control the size n' of the approximating matrices in terms of the size n of the original matrices, ε and δ.

References

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